Extending the Quasidifferential Framework: From Fixed-Key to Expected Differential Probability

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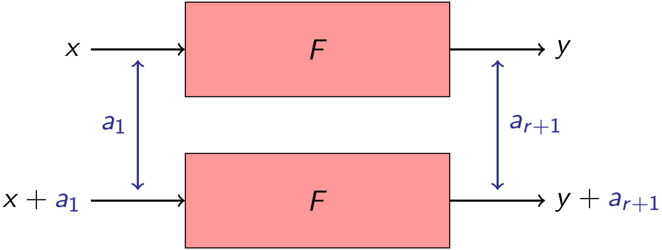
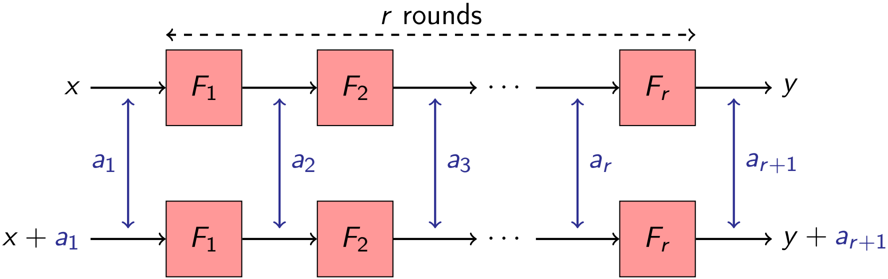
Differential Cryptanalysis

- Introduced by Biham and Shamir in 1990 [BS91].



Distinguisher: differential (a_1, a_{r+1}) such that $\Pr[a_1 \to a_{r+1}] \gg \frac{1}{2^n}$.

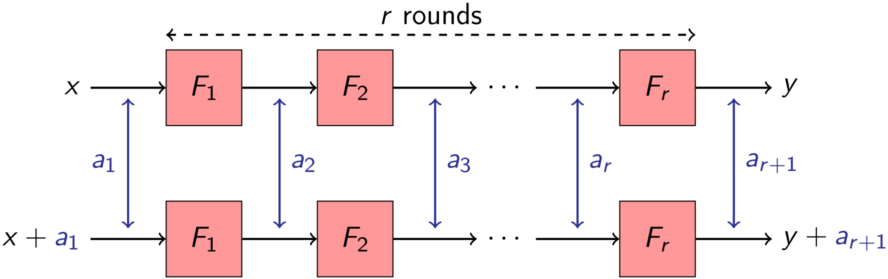
Differential Characteristics



Distinguisher probability estimation: characteristic $(a_1, a_2, \ldots, a_{r+1})$ such that

$$\Pr[a_1 \to a_{r+1}] \geq \Pr[a_1 \to a_2 \to \cdots \to a_{r+1}] \gg \frac{1}{2^n}.$$

Differential Characteristics



Distinguisher probability estimation: characteristic $(a_1, a_2, \ldots, a_{r+1})$ such that the fixed-key probability verifies $\Pr_k[a_1 \to a_2 \to \cdots \to a_{r+1}] \gg \frac{1}{2^n}$ for any given key.

Classical Assumptions

Stochastic Equivalence Hypothesis

$$\mathsf{Pr}_k[a_1 \to a_2 \to \cdots \to a_{r+1}] \approx \underbrace{\frac{1}{|\mathcal{K}|}\sum_{k\in\mathcal{K}} \mathsf{Pr}_k[a_1 \to a_2 \to \cdots \to a_{r+1}]}_{\text{Expected Differential Probability}} \quad \forall k \in \mathcal{K}$$

Round Independence

$$\mathrm{EDP}[a_1, \ldots, a_{r+1}] \approx \prod_{i=1}^{r} \mathsf{Pr}[a_i \to a_{i+1}]$$

Reasonable Hypotheses?

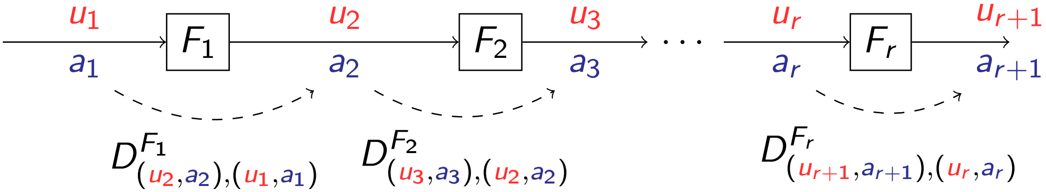
- Deviation of the fixed-key probability already observed by Knudsen in 1992 [Knu93].
- Most of AES characteristics are plateau characteristics (up to 4 rounds) [DR07].
- Only **1 out of 43** published characteristics on SKINNY is valid for more than 50% of the keys [PT22].

[BR22]'s quasidifferential framework:

$$p_k = \prod_{i=1}^{r} \left(\frac{1120}{64^3} - (-1)^{k_{2i,12}+k_{2i,14}} \frac{672}{64^3} \right)$$

They provide a general framework to evaluate the fixed-key probability.

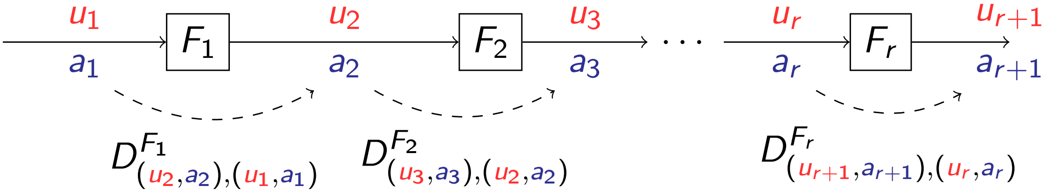
Quasidifferential Framework



where

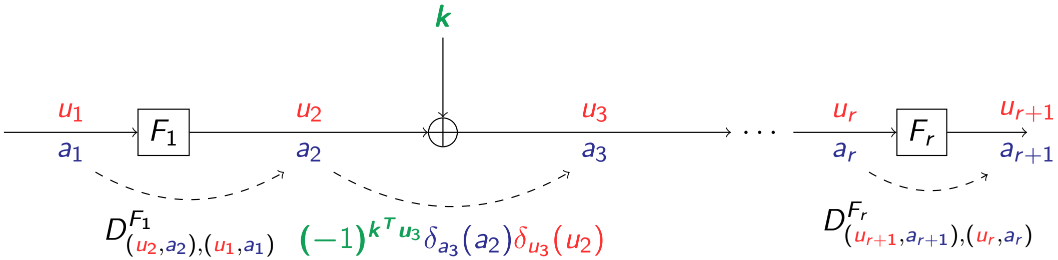
$$
\begin{aligned}
D^{F_i}_{(u_{i+1},a_{i+1}),(u_i,a_i)} = &\left(2\Pr[u_{i+1}^{\mathsf{T}} F_i(\boldsymbol{x}) \oplus u_i^{\mathsf{T}} \boldsymbol{x} = 0 \mid F_i(\boldsymbol{x} \oplus a_i) \oplus F_i(\boldsymbol{x}) = a_{i+1}] - 1\right) \\
&\times \Pr[F_i(\boldsymbol{x} \oplus a_i) \oplus F_i(\boldsymbol{x}) = a_{i+1}]
\end{aligned}
$$

Quasidifferential Framework



$$\operatorname{Corr}\Big((u_1, a_1), \ldots, (u_{r+1}, a_{r+1})\Big) = \prod_{i=1}^{r} D^{F_i}_{(u_{i+1},a_{i+1}),(u_i,a_i)}$$

Quasidifferential Framework - Key Addition



$$\mathrm{Corr} = D^{F_1}_{(u_2,a_2),(u_1,a_1)} \times (-1)^{k^T u_3}\delta_{a_3}(a_2)\delta_{u_3}(u_2) \times \cdots \times D^{F_r}_{(u_{r+1},a_{r+1}),(u_r,a_r)}$$

Fixed-key Probability As Sum Of Correlations

Theorem 4.1 [BR22]

$$\Pr_k\left[\bigwedge_{i=1}^{r} F_i(\boldsymbol{x}_i \oplus a_i) \oplus F_i(\boldsymbol{x}_i) = a_{i+1}\right] = \sum_{u_2,\ldots,u_r} \prod_{i=1}^{r} D^{F_i}_{(u_{i+1},a_{i+1}),(u_i,a_i)}$$

with $u_1 = u_{r+1} = 0$, $\boldsymbol{x}_i = F_{i-1}(\boldsymbol{x}_{i-1})$, $\boldsymbol{x}_1$ uniform.

No assumptions needed!

Our Contributions

Related-key setting

The original framework applies the same function on both elements of the pairs. Not compatible with the related-key setting.

We extend the original framework to treat pairs asymmetrically.

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Analysing clusters of differential characteristics

- Exhausting all quasidifferential trails for a characteristic can be hard or infeasible.
- Analysing cluster leads to complex formulas which can be heavy to manipulate.

- Extend [BR22] framework to obtain an **exact formula for the EDP**.
- Takes the key-schedule into account for the first time!
- Always faster than fixed-key analysis.

Extension 1: Related-key Setting

Standard basis

$$T^F \otimes T^F$$

where $T^F : \delta_x \mapsto \delta_{F(x)}$

$\updownarrow$ change-of-basis

Quasidifferential basis

$$D^F = \mathcal{Q}_m (T^F \otimes T^F) \mathcal{Q}_n^{-1}$$

Elements of the pairs are treated symmetrically.

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Problem: Key addition in related-key setting is asymmetric.

Extension 1: Related-key Setting

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- Similar theorems as [BR22] can be derived.
- Applicable to related-key setting without additional complexity.

Let $F : x \mapsto x + k,\ G : x \mapsto x + k + cst$.

$$D^{F/G}_{(v,b),(u,a)} = (-1)^k \delta_v(u) \delta_b(a + cst)$$

Extension 2: Exact EDP Computation

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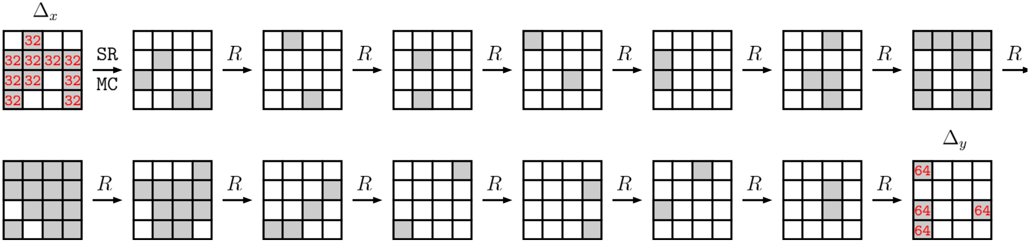

Figure: Truncated differential trail of the attack on 25-round `SKINNY-128-384` [BDD+23]

Extension 2: Exact EDP Computation

What really does the quasidifferential framework?

Given a sequence of operations $(F_1, \ldots, F_r)$ we can derive the probability of a characteristic over all possible inputs as a function parameterized by the key.

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What if we consider the key as input?

$$x \mapsto F(x)$$

$$D^F_{(v_x,b_x),(u_x,a_x)}$$

change of perspective $\longleftrightarrow$

$$(x, k) \mapsto \overbrace{(F(x), k)}^{:=\overline{F}(x,k)}$$

$$D^{\overline{F}}_{(v_x||v_k,b_x||b_k),(u_x||u_k,a_x||a_k)}$$

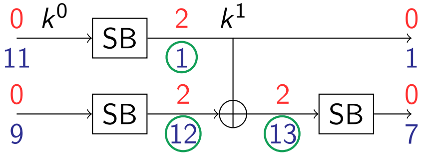
$$= D^F_{(v_x,b_x),(u_x,a_x)}\delta_{v_k}(u_k)\delta_{b_k}(a_k)$$

Key Addition

Key Addition

Let $G : (x, k) \mapsto (x + k, k)$. The masks behave as follows:

$$D^G_{(v_x||v_k, v_x||b_k),(v_x||u_k, v_x||a_k)} = \delta_{b_x}(a_x + a_k)\delta_{v_x}(u_x)\delta_{b_k}(a_k)\delta_{v_k}(u_x + u_k)$$



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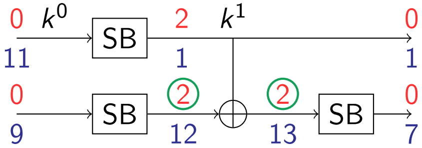

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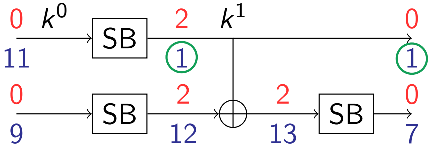

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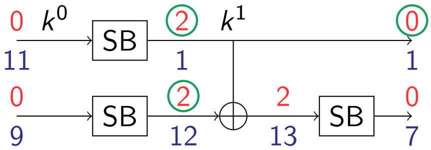

EDP As Sum Of Correlations

EDP Exact Formula

Let $E = F_r \circ \cdots \circ F_1$ and let $Q = ((a_x^1, a_k^1), \ldots, (a_x^{r+1}, a_k^{r+1}))$ represent a differential characteristic over E. Then,

$$EDP(Q) := \Pr\left[\bigwedge_{i=1}^{r} F_i\left((\boldsymbol{x_i}, \boldsymbol{k_i}) + (a_x^i, a_k^i)\right) + F_i(\boldsymbol{x_i}, \boldsymbol{k_i}) = (a_x^{i+1}, a_k^{i+1})\right]$$

$$= \sum_{\substack{u_x^2, \ldots, u_x^r \\ u_k^2, \ldots, u_k^r}} \prod_{i=1}^{r} D^{F_i}_{((u_x^{i+1}, u_k^{i+1}), (a_x^{i+1}, a_k^{i+1})), ((u_x^i, u_k^i), (a_x^i, a_k^i))}$$

where $(\boldsymbol{u_x^1}, \boldsymbol{u_k^1}) = (\boldsymbol{u_x^{r+1}}, \boldsymbol{u_k^{r+1}}) = (0, 0)$, $(\boldsymbol{x_i}, \boldsymbol{k_i}) = F_{i-1}(\boldsymbol{x_{i-1}}, \boldsymbol{k_{i-1}})$ for $i = 2, \ldots, r$ and $(\boldsymbol{x}_1, \boldsymbol{k}_1)$ uniformly random on $\mathbb{F}_2^n \times \mathbb{F}_2^n$.

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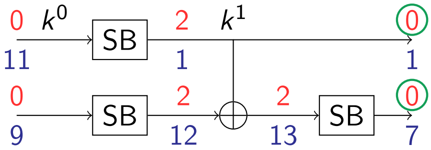

Finding Few Trails: Intuition

- Quasidifferential trail without the key: explains local effect on some keys.
- Quasidifferential trail with the key: explains global effect on all keys.

Applications: AES and SKINNY

Developed a practical MILP implementation to search for quasidifferential trails.

AES: EDP matches the heuristical estimation.

Version	Rounds	Estimated proba.	EDP	Source
AES-128	2	2^{-7}	2^{-7}	[FJP13]
AES-128	4	2^{-81}	2^{-81}	[FJP13]
AES-128	4	2^{-81}	2^{-81}	[FJP13]
AES-128	5	2^{-105}	2^{-105}	[FJP13]
AES-256	14	2^{-154}	2^{-154}	[GLMS18]
AES-256	14	2^{-146}	2^{-146}	[GLMS18]
AES-192	9	2^{-146}	2^{-146}	[GLMS18]

[1] [FJP13] Fouque et al. *Structural Evaluation of AES and Chosen-Key Distinguisher of 9-round AES-128.* CRYPTO 2013

[2] [GLMS18] Gérault et al. *Revisiting AES Related-Key Differential Attacks with Constraint Programming.* Inf. Process. Lett. 2018

Applications: AES and SKINNY

SKINNY: More precise results than Peyrin and Tan on SKINNY-64 in fixed-key model.
More accurate estimation of EDP.

SKINNY	Estimated prob.	EDP	[PT22] Key Space	[PT22] Prob. Range
64-64	2^{-52} 2^{-46}	2^{-52} (1) 0 (8)	2^{-6} 0	2^{-46} —
64-128	2^{-55} 2^{-44}	2^{-55} (1) 2^{-44} (4)	2^{-4} Not given	2^{-51} 2^{-39} – $2^{-35.415}$
64-192	2^{-54}	2^{-54} (1)	$2^{-6.19}$	2^{-48} – 2^{-47}
128-128	2^{-123} 2^{-120}	0 (16) $2^{-119.05}$ **(44)**	0 $2^{-7.66}$	— $2^{-122.39}$ – $2^{-106.88}$ (E)
128-256	$2^{-127.66}$	$2^{-126.41}$ **(26)**	$2^{-6.11}$	$2^{-133.80}$ – $2^{-112.15}$(E)

Applications: AES and SKINNY

SKINNY: Analysed a cluster of 114 688 characteristics with EDP computation:
More than a half of the characteristics are invalid.
Improved Diff-MitM attack on SKINNY-128-384 by a factor $2^{2.9}$.

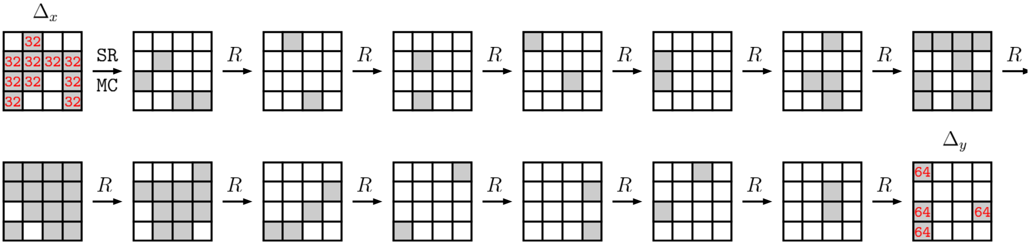

Figure: Truncated differential trail of the attack on 25-round SKINNY-128-384 [BDD+23]

[1] [BDD+23] Boura et al. *Differential meet-in-the-middle cryptanalysis.* CRYPTO 2023

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Summary

- Extend the quasidifferential framework to the related-key setting.
- Provide for the **first time** a formula for the EDP that takes the key-schedule into account.
- Practical MILP model.
- New results on `AES` and `SKINNY`.

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Thanks for your attention!

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